Defending the CAPM

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Abstract

A capital asset pricing model is devised to defend the security market line results of Sharpe (1964), Lintner (1965). and Mossin (1966) from the attacks of Fama and French (1992, 1996). The model of this paper does not assume a particular utility function, knowledge of the quantity of investor wealth, or the application by investors of a particular probability density function. Moreover, this model works in the presence of heterogeneous investor expectations and incomplete markets. Also, it is not required that the market portfolio be measured in order to value assets.

Introduction

Can the CAPM be saved? ask Fama and French (1992, 1996) as they reflect upon empirical difficulties associated with comparing model predictions to actual outcomes. If not by way of data analysis, there remains hope by way of theorem-and-proof. Below are three propositions, each more general than CAPM in the sense that these propositions work in reference to any portfolio of a certain kind. The strategy to save the CAPM is to describe simply with a small number of believable assumptions how assets are priced in the contexts of all of these portfolios of a certain kind. And then move to the specific case of asset pricing in the context of one portfolio of that kind, the market portfolio. In this way one can move from talking about what must be true for portfolios that investors can actually measure to talking about what must be true for the market portfolio, which no investor can measure.

Propositions

The problem of this paper is to devise a useful capital asset pricing theory under the assumption that empirical testing of all capital asset pricing models is impossible. Thus the positive economics of Freidman (1953) is abandoned. As such, care is taken with assumptions employed in the logical arguments found below. Unlike much of the earlier work in capital markets theory, these present arguments do not attempt to describe what particular utility function drives investors in their decision making or what particular probability density function they use to describe possible future values. Such assumption statements are essential for the economist who wishes to know the connection of utility theory to investment decision making and for the econometrician who wished to describe data in a certain way, with only means and variances perhaps. But these kinds of assumptions may not be good descriptions of reality, and for this reason their use is avoided in the arguments of this paper.

Assumption statements requiring knowledge of the quantity of investor wealth, heterogeneous investor expectations, and complete markets are avoided on these same grounds. Also it is not assumed that investors measure things that they cannot measure, rates of return on the market portfolio, changes in aggregate consumption or other market-wide factors.

How does the pricing model of this paper stand in relation to the CAPM? Certainly the assumptions employed here are different than those employed by Sharpe (1964), Lintner (1965), and Mossin (1966). And it is also true that this model is more general than CAPM, for the model of this paper works with or without the market portfolio. But nothing here contradicts CAPM. Furthermore, essential properties of this model are identical with those of CAPM. Asset returns are a simple linear function of a risk free rate and market portfolio returns; and asset returns that are uncorrelated with market portfolio returns are not priced.

Proposition I:

Identify a portfolio such that all possible end of period future portfolio values are greater than zero. In the context of portfolios like this, the law of one price holds if and only if the equation below correctly values each of the assets contained within this portfolio.

$$V[i] = \sum_s \frac{p(s)F[i,s]}{r[s]} \qquad (1)$$

Where: V[i] is the current market value of a long position in asset i which is contained within a portfolio with present market value equal to V; F[i, s] is the future value in state s of a long position in asset i.; p(s) is a joint probability describing the likelihood of a future state of the world being realized at time period one. This state obtains if and only if asset i generates a price relative r [i, s] and the chosen portfolio V which contains asset i generates a price relative r[s]. Proof: see Appendix.

Proposition II:

To the assumptions of Proposition I (There exists a portfolio such that all state-contingent portfolio future values are greater than zero, and the law of one price holds.) add an assumption that investors can short sell predicted state-contingent future values such that these that errors in these predictions are mean zero and independent of actual end of period outcomes. With these assumptions in mind: unpredicted returns are not priced if and only if the present market value of assets contained within a portfolio may be stated in terms of possible predicted future values:

$$V[i] = \sum_s \frac{p(s)\hat{F}[i,s]}{r[s]} \qquad (2)$$

Where $\hat{F}[i,s]$ designates the predicted end-of-period cash flow for asset i in future state s is realized. Proof: see Appendix.

Proposition III:

Make the assumptions of Proposition II (There exists a portfolio such that all state-contingent portfolio future values are greater than zero, and the law of one price holds; investors can short sell predicted state-contingent future values such that these that errors in these predictions are mean zero and independent of actual outcomes.) and to these add that there exists an asset which pays a risk free rate. In this context, the predicted state-contingent price relatives of assets contained within a portfolio are simple linear functions of the risk free price relative and the state-contingent price relatives of the portfolio to which they belong if and only if the equation below holds for every asset i

$$V[i] = \sum_s \frac{p(s)F[i,s]}{r[s]} \qquad (1)$$

Where: V [i] is the current market value of a long position in asset i, $F[i,s]$ is the future value in state s of a long position in asset i, p (s) is a joint probability describing the likelihood of a future state of the world being realized at time period one: this state obtains if and only if asset i generates a price relative r [i, s] and the given portfolio V generates a price relative r [s]. These predicted price relatives need not depend upon the ability to measure possible rates of return on the market portfolio. Other portfolios may be substituted so long as epsilon risk can be diversified away. Proof: see Appendix.

Should one believe Proposition I is a realistic description of investor behavior in capital markets. Denial of this proposition entails a belief that the law of one price does not hold and/or a belief that one cannot find a portfolio that must in all possible states of nature that could obtain one period from now have a strictly positive future value. Perhaps there are states of nature where one or both of these denials are plausible. In these states of nature Proposition I loses its logical force. However, there are other states of nature including I would suppose many practical, real-world scenarios where such denials must be mistaken and in every one of these states Proposition I must be true.

Proposition I does not claim that investors predict asset returns or portfolio returns. Proposition II says that to do so, it must be the case that unpredictable returns are not priced, and *vice versa*: if unpredictable returns are priced, then investor predictions are unconnected to prices. Proposition II is general to the degree that it does not specify what tools investors might use to predict future returns, and it does not require that investors agree as to what predictions should be made..

Proposition III shows a logical equivalence between making predictions of asset returns which are linear in portfolio returns and the existence of a marketplace where unpredictable returns are not priced. This logical equivalence permits investors to describe investment returns as a simple linear regression of returns on any portfolio that must have a strictly positive future value so long as unpredictable returns are not priced. So Proposition III is a bi-conditional statement: the ability to diversify away all unsystematic risk is the guarantor of linearity and *vice versa* B a simple linear relationship of the predicted returns of some asset to portfolio returns guarantees that uncorrelated returns are not priced. If the portfolio under analysis is the market portfolio, then the simple linear relationship above is the security market line.

Given that the distinction between diversifiable and non-diversifiable risk and that the security market line are essential CAPM properties, denial of Proposition III is then a denial of CAPM. And if one maintains that the only essential properties of CAPM are the security market line and the distinction between systematic and unsystematic risk then one must maintain that an affirmation of Proposition III is an affirmation of the CAPM.

Connections to a Modern Controversy

In response to the Fama and French (1992) empirical criticism of the CAPM, Black (1993) makes the following observation: if the returns of low-beta stocks are similar to that of high-beta stocks, then the CAPM is not in fact useless, for investors should purchase low-beta stocks and sell high-beta stocks. But there are problems with this view. This observation of Black's (1993) implicitly assumes that investors can either measure CAPM beta, an assumption put into dispute both by Roll's (1977) critique of the CAPM and by the Fama and French (1996) paper, or that betas calculated by using portfolios other than the market portfolio as a benchmark are adequate measures of risk and return, a position unsubstantiated by the CAPM that Black (1993) defends.

The asset pricing model of this paper works on subsets of the market portfolio and the market portfolio too, thereby permitting investors to compare betas calculated using a proxy for the market folio with CAPM betas. With this achievement in hand, one could then follow Black's (1993) advice. Consider an investment in some asset i. What if this asset were held in two different portfolios? No matter the different portfolio context, all must agree that the expected price relative of this asset would be the same in either portfolio. Let one of these portfolios be the market portfolio, m, and let the other portfolio be a subset of that market portfolio. First, write the expected price relative of asset i as equal to the expected price relative of a two asset portfolio consisting of a risk free asset and a chosen subset portfolio.

$$E(r[i]) = r(1 - h[i]) + h[i]E(r[s]) \qquad (3)$$

Where: E(r[i]) is the expected price relative of asset i, r is the risk free price relative, E(r[s]) is the expected price relative of a chosen subset portfolio, (1-h[i]) and h[i] are portfolio weights chosen to produce the expected price relative of asset i. Second, consider asset i in the context of the market portfolio.

$$E(r[i]) = r(1 - b[i]) + b[i]E(r[m]) \qquad (4)$$

Where: E(r[i]) is the expected price relative of asset i, r is the risk free price relative, E(r[m]) is the expected price relative of the market portfolio, and (1-b[i]) and b[i] are portfolio weights chosen to produce the expected price relative of asset i. Given r and one of the following parameters: E(r[i]), E(r[s]), E(r[m]), one may find all portfolio weights. And, if the CAPM is true, then the portfolio weighting solution in equation (4) above is also a solution for CAPM beta.

This solution avoids the problem of Roll's (1977) critique of the CAPM. Knowledge of a subset of the market portfolio can be used to infer expected return on the market portfolio and CAPM beta. A problem of mean – variance inefficiency comes into play if and only if the law of one price is violated -- if and only if the expected return on asset i changes as portfolio context changes.

Connections to an Ancient Problem

The general pricing model of this paper, and by extension the CAPM, can be viewed as an option pricing model. To see this consider an ancient problem that still inspires comment and analysis to this present day. In the year 1738, Daniel Bernoulli imagines a merchant who lives in St. Petersburg awaiting a shipment of commodities now at sea. The present market value of these commodities ashore, out of the reach of pirates and storms, is equal to V. The gamble to purchase these commodities and ship them by sea is risky for two reasons. Because of pirates and storms there is a significant probability the merchant will never receive this cargo. Moreover, the value of these commodities, presently in the hold of a ship at sea, fluctuates as the health and wellbeing of the overall economy changes.

The probability that the voyage ends in disaster, ship and cargo lost, is equal to p [z]. This is a subjective probability estimate of investor z, not a risk-neutral probability produced from the maintenance of a risk-free hedge nor a probability number associated with a composite investor existing in the imagination of financial economists searching for a solution to the aggregation problem. What will investor z be willing to pay for these commodities now at sea?

$$G[z] = V(1 - p[z]) \qquad (5)$$

G [z] is symbolic of investor z's estimate of the intrinsic present market value of commodities now at sea. This solution is a specific application of Proposition I. which in this application assumes that the commodities at sea could never have a zero end-of-period future value were they safely in the hands of the merchant who had ordered them.

In addition to maritime transport there are other capital budgeting applications. Consider the capital budgeting problems of drilling oil wells, putting satellites in orbit, or those of crop selection. In all of these endeavors the source of risk is two-fold. First, the prize itself may be a risky asset, its present market value, V, dependent upon market forces. Such is the case with oil and natural gas, satellite rents, and harvests. And second, it may happen that the prize which is sought for is lost. The oil well may be a dry hole, the satellite may not reach the proper orbit, and the crop may never be harvested. There is a CAPM connection. These commodities at sea could instead the market portfolio. Instead of pirates, an all-or-nothing call option where the owner of the investment with price G receives the market portfolio at expiry if and only if its value at expiry exceeds a striking price.

Conclusion

The CAPM is saved. A security market line can be produced by analysis of subsets of the market portfolio. Proxies for these subsets are unnecessary thus the criticism of Roll (1977) and of Fama and French (1992, 1996) need not be persuasive in the condemnation of CAPM. Application of the general pricing model of this paper to the specific case of the CAPM shows that the security market line can exist if and only if unsystematic risk is not priced. If one agrees that the security market line and the distinction between systematic and unsystematic risk are the two essential CAPM properties, then one must also agree that denial of CAPM entails denial of the distinction between systematic and unsystematic risk. Such a denial is implausible. QED.

Appendix

Proposition I: Identify a portfolio such that all possible end of period future portfolio values are greater than zero. In the context of portfolios like this, the law of one price holds if and only if the equation below correctly values each of the assets contained within this portfolio.

$$V[i] = \sum_{s} \frac{p(s)F[i,s]}{r[s]} \qquad \text{(A.1)}$$

Where: V[i] is the current market value of a long position in asset i which is contained within a portfolio with present market value equal to V; F [i, s] is the future value in state s of a long position in asset i, p (s) is a joint probability describing the likelihood of a future state of the world being realized at time period one: this state obtains if and only if asset i generates a price relative r [i, s] and the given portfolio which contains asset i generates a price relative r [s]. It is not assumed that investors hold these probability estimates in common or that these estimates are produced are an average of probability estimates held by a group of investments.

Proof of necessity: by hypothesis the law of one price holds, therefore, the future value of any portfolio is always equal to the sum of the future values of all the assets contained with that portfolio. For all assets included in a portfolio, i = 1, .., n, for all possible states, s, and for all portfolio weights, w [1], ...,w [n], the price relative of every possible portfolio may be expressed as follows:

$$r[s] = r[1,s]w[1] + \ldots + r[n,s]w[n] \qquad \text{(A.2)}$$

The only choice variables for investors are asset weights. These decisions, w [1], ..,w [n], are made as a portfolio is purchased (sold). For all investors and for all assets included in any portfolio, the sum of all these asset weights must be equal to unity. By assumption one considers only portfolios such that all possible portfolio future values are greater than zero. This being so, r [s] is greater than zero so one may divide r [s] by itself. Therefore, for any eligible portfolio, for any state, for every investor:

$$1=\frac{r[1,s]w[1]}{r[s]}+\ldots+\frac{r[n,s]w[n]}{r[s]} \qquad \text{(A.3)}$$

Multiplication of this result above by the joint probability which describes the likelihood that state s occurs is shown below.

$$p(s)=\frac{p(s)r[1,s]w[1]}{r[s]}+\ldots+\frac{p(s)r[n,s]w[n]}{r[s]} \qquad \text{(A.4)}$$

Appearing in the numerator of the expression above are two positive numbers: p(s), a probability number, which is greater than or equal to zero of necessity, and also r (i, s), the state-contingent price relative for any asset i contained within a portfolio which is a positive number if possible investor losses are limited to 100 percent of investment value. Note that each possible future state s is mutually exclusive of every other possible state s, and the list of these possible states is exhaustive. Therefore, applying the law of total probability, the sum of all these probabilities is equal to the real number one. Summing over all possible states produces a convex combination which is described below:

$$1=w[1]\sum_s\frac{p(s)r[1,s]}{r[s]}+\ldots+w[n]\sum_s\frac{p(s)r[n,s]}{r[s]} \qquad \text{(A.5)}$$

Given the definition of a given portfolio weight as the ratio of the value of a given asset to the value of some portfolio that contains this given asset, multiplication of both sides of the equation above by the current market value V of a portfolio provides the following valuation equation.

$$V=V[1]\sum_s\frac{p(s)r[1,s]}{r[s]}+\ldots+V[n]\sum_s\frac{p(s)r[n,s]}{r[s]} \qquad \text{(A.6)}$$

Where V [1] is the current market value of investment in asset 1, and V [n] is the current market value of investment in asset n. By hypothesis, under the law of one price it is also true that the present market value of any portfolio must equal the sum of the present market value of all asset positions. Also it may be said that the law of one price holds if and only if the sum of the present market values of assets contained with a portfolio is equal to the present market value of that portfolio. Thus equation (A.7) below is logically equivalent to a claim that the law of one price holds.

$$V=V[1]+\ldots+V[n] \qquad \text{(A.7)}$$

The left-hand side of both equations (A.6) and (A.7) above being equal, one may then combine these two equations to produce the following convex combination described as (A.8). This equation below (A.8) may also be interpreted as a zero cost portfolio that is produced by simultaneously taking a long position in portfolio V which is described by equation (A.6) and a short position in that same portfolio V which is described in equation (A.7):

$$0=\left(\left(V[1]\sum_{s}\frac{p(s)r[1,s]}{r[s]}-V[1]\right)+\ldots+\left(V[n]\sum_{s}\frac{p(s)r[n,s]}{r[s]}-V[n]\right)\right) \qquad \text{(A.8)}$$

The zero cost portfolio above consists of long and short positions in i = 1, …, n assets. If the law of one price holds, for each of these assets the value of the long holding is equal to that of the short holding. Consider any investor who is willing to pay for a position in asset i a value equal to I [i] for asset i contained within a portfolio V. Let the equation below define this quantity.

$$I[i]=V[i]\sum_{s}\frac{p(s)r[i,s]}{r[s]} \qquad \text{(A.9)}$$

Substitution of this definition contained in (A.9) into (A.8) provides the following:

$$0=(I[1]-V[1])+\ldots+(I[n]-V[n]) \qquad \text{(A.10)}$$

If the law of one price holds for every asset i, only if I [i] is equal to V [i] for all assets contained within any portfolio. It follows from this that for any asset i contained within portfolio V

$$1=\sum_{s}\frac{p(s)r[i,s]}{r[s]} \qquad \text{(A.11)}$$

By definition, the state-contingent, end-of-period future value of asset i is equal to the present market value of a position in asset i multiplied by the state-contingent price relative of asset i:

$$F[i,s]=V[i]r[i,s] \qquad \text{(A.12)}$$

With this definition in mind, one may write the present market value of a position in asset i:

$$V[i]=\sum_{s}\frac{p(s)F[i,s]}{r[s]} \qquad \text{(A.1)}$$

This proof of necessity is ended.

Proof of sufficiency: by the hypothesis of this proof of sufficiency, the present market value of a position in asset i is equal to the following

$$V[i]=\sum_{s}\frac{p(s)F[i,s]}{r[s]} \qquad \text{(A.1)}$$

With this equation of asset valuation in mind, consider a collection of n assets that contains an asset i:

$$Y = \sum_s \frac{p(s)F[1,s]}{r[s]} + \ldots + \sum_s \frac{p(s)F[n,s]}{r[s]} \qquad \text{(A.13)}$$

where Y is a quantity equal to the sum of the right-hand side of equation (A.13). Consider this same portfolio described in equation (A.13) above again in equation (A.7) written below, which says that the present market value of a portfolio is equal to the sum of the values of the assets contained within that portfolio.

$$V = V[1] + \ldots + V[n] \qquad \text{(A.7)}$$

Let an investor take both a long and short position. The result is shown below.

$$Y - V = \left(\left(\sum_s \frac{p(s)F[1,s]}{r[s]} - V[1] \right) + \ldots + \left(\sum_s \frac{p(s)F[n,s]}{r[s]} - V[n] \right) \right) \qquad \text{(A.8)}$$

By hypothesis of this proof of sufficiency, for each asset i of the portfolio described in equation (A.13) the current market value is equal to V[i]. So then, Y is equal to V. Thus, the present market value of a portfolio is equal to the sum of the present market values of assets contained within that portfolio: the law of one price holds. This ends this proof of sufficiency. Proofs of necessity and sufficiency having been accomplished, this argument is then complete. Q.E.D.

Proposition II: To the assumptions of Proposition I (There exists a portfolio such that all state-contingent portfolio future values are greater than zero, and the law of one price holds.) add an assumption that investors can short sell predicted state-contingent future values such that these that errors in these predictions are mean zero and independent of actual end of period outcomes. With these assumptions in mind: unpredicted returns are not priced if and only if the present market value of assets contained within a portfolio may be stated in terms of possible predicted future values:

$$V[i] = \sum_s \frac{p(s)\hat{F}[i,s]}{r[s]} \qquad \text{(A.14)}$$

Where $\hat{F}[i,s]$ designates the predicted end-of-period cash flow for asset i in future state s is realized.

Proof of necessity: This argument begins with what is essential to the notion of prediction. Consider an investor who simultaneously takes a long position in asset i costing one unit of currency and a short position in the predicted future value of asset i. This short position results in a present cash inflow to the investor equal to one unit of currency. End-of-period payoff to this epsilon portfolio is described in the equation below.

$$\varepsilon[i,s] = r[i,s] - \hat{r}[i,s] \qquad \text{(A.15)}$$

Where symbol ε [i, s] has two meanings: it is the state-contingent future value of the portfolio described above, and it is also describes the forecasting error associated with asset i in state s, and the Ahat@ superscript designates prediction. By construction, each epsilon portfolio associated with some asset i has three essential properties. First, for each asset i contained with a portfolio, states are partitioned such that there is only one possible future value in state s and only one predicted price relative of asset i of in that same state. Second, the expected future value of an epsilon portfolio is equal to zero. Third, all possible future values of an epsilon portfolio are uncorrelated with all possible future values of a portfolio containing asset i and uncorrelated with the possible future values of asset i. Given these essential properties of an epsilon portfolio, one may write the present value of all possible returns on investment in any asset i which is included in a portfolio in the following manner.

$$\sum_s \frac{p(s)r[i,s]}{r[s]} = \sum_s \frac{p(s)\hat{r}[i,s]}{r[s]} + \sum_s \frac{p(s)\varepsilon[i,s]}{r[s]} \qquad \text{(A.16)}$$

Recall that r [s] is strictly greater than zero for all portfolios under consideration. The present value of all possible future price relatives for asset i are equal to the present value of all possible predicted price relatives associated with asset i plus the present value of all possible epsilon portfolio returns associated with asset i. By construction the possible future values of an epsilon portfolio corresponding to same asset i are uncorrelated with the possible future values of asset i and uncorrelated with the possible future values of the portfolio containing asset i. This being so, the present value of an epsilon portfolio associated with asset i in the following manner:

$$\sum_s \frac{p(s)\varepsilon[i,s]}{r[s]} = \sum_s \frac{p(s)}{r[s]} \cdot E(\varepsilon[i,s]) \qquad \text{(A.17)}$$

where the expectation of the future returns on an epsilon portfolio associated with asset i is written E (ε [i, s]). Again by construction, this expected future value of the epsilon portfolio must equal zero.

$$E(\varepsilon[i,s]) = 0 \qquad \text{(A.18)}$$

Therefore, it follows immediately from the essential properties of epsilon portfolios that the present market value of all possible realized price relatives is equal to the present value of all predicted price relatives:

$$\sum_s \frac{p(s)r[i,s]}{r[s]} = \sum_s \frac{p(s)\hat{r}[i,s]}{r[s]} \qquad \text{(A.19)}$$

By definition, the end-of-period predicted cash flow of asset i in state s is equal to the product of the current price of asset i and the state contingent predicted price relative of asset i:

$$\hat{F}[i,s] = V[i]\, r[i,s] \qquad \text{(A.20)}$$

Simple algebra produces the desired conclusion described in equation (A.14). This proof of necessity is ended.

Proof of sufficiency: By hypothesis of this proof of sufficiency

$$V[i]=\sum_{s}\frac{p(s)\hat{F}[i,s]}{r[s]} \qquad \text{(A.14)}$$

Given the law of one price and that the portfolio under consideration must always produce an end-of-period future value that is strictly positive, assumptions employed both to produce Proposition I and serve in this present argument, one may again write Proposition I:

$$V[i]=\sum_{s}\frac{p(s)F[i,s]}{r[s]} \qquad \text{(A.1)}$$

Setting equations (A.14) equal to (A.1) entails the following equation:

$$\sum_{s}\frac{p(s)r[i,s]}{r[s]}=\sum_{s}\frac{p(s)\hat{r}[i,s]}{r[s]} \qquad \text{(A.19)}$$

The notion of prediction demands that for asset i in each state s, there is only one possible return and only one predicted return. Therefore, one may rewrite equation (A.19) above as follows:

$$0=\sum_{s}\frac{p(s)(r[i,s]-\hat{r}[i,s])}{r[s]} \qquad \text{(A.21)}$$

Substitution reflecting the definition of epsilon permits the rewriting of equation (A.22):

$$0=\sum_{s}\frac{p(s)\varepsilon[i,s]}{r[s]} \qquad \text{(A.22)}$$

Thus the present market value of unpredicted returns is equal to zero. This proof of sufficiency is ended. Proofs of necessity and sufficiency are complete. Q.E.D.

Lemma. Given the law of one price, the existence of a portfolio which has a strictly positive future value in all possible future states-of-nature, and given that within this portfolio there exists an asset which pays a constant, strictly positive future value one period from now in every possible state-of-nature; it is then necessary that:

$$1=r\sum_{s}\frac{p(s)}{r[s]} \qquad \text{(A.23)}$$

Where: r, is the price relative of a risk-free asset; r [s] is the price relative of a portfolio contingent upon state s being realized, p (s) is the probability that state s is realized. Given the law of one price, this risk free rate must be unique B there can not be more than one risk free rate in a market. And, by the law of one price, every risk free investment must earn this same risk free rate.

Proof: By hypothesis, let there exist a portfolio such that it is always true that the state-contingent portfolio future value is greater than zero one period from now. I also assume that this portfolio also contains a risk free asset. This risk free asset is designated as asset n.

Therefore, the present market value of this portfolio may be written as:

$$V = V\left(w[1]\sum_s \frac{p(s)r[1,s]}{r[s]} + ... + w[n]\sum_s \frac{p(s)r[n,s]}{r[s]} \right) \qquad \text{(A.24)}$$

for all assets contained within this portfolio. Where: V is the current market value of a given portfolio; p (s) is a joint probability that describes the likelihood of asset i attaining a specific price relative corresponding to state s while at the same time the portfolio attains a specific price relative: r [i, s] is a possible price relative which is attained by asset i if state s occurs one period from the date of investment; and r [s] is a possible price relative which is attained by a portfolio V if state s is realized one period from the date of investment. This equation above is similar to equation (A.6) of Proposition I above. The only difference is the assumed existence of a risk free asset contained within a portfolio. One may describe the holdings of this portfolio described in the equation (A.24) above as belonging in one of two categories. The first category contains only risky assets and the other category contains only a position in a risk free asset. This ability permits one to write the equation below which shows the present value of the risk and the risk free categories separately. The first term of the right-hand side of the equation below is the risky portion of a given portfolio and the second term of the right-hand side of the equation below is the risk free portion of a given portfolio. Recall, asset n is designated a risk free asset.

$$V = V\left(w[1]\sum_s \frac{p(s)r[1,s]}{r[s]} + ... + w[n-1]\sum_s \frac{p(s)r[n-1,s]}{r[s]} \right) + V\left(w[n]\sum_s \frac{p(s)r}{r[s]} \right) \qquad \text{(A.25)}$$

where r is equal to the risk free price relative. In the context of this given portfolio, the present value of risk free investment is, by construction, equal to the product of the asset weight which denotes investment in a risk free asset and the present value of the entire portfolio that is being considered.

$$R = V \cdot w[n] \qquad \text{(A.26)}$$

Where: R is the present market value of risk free investment found in a given portfolio; V is the current market value of investment in that portfolio, and w [n] is the asset weight denoting the relative size of risk free investment in that portfolio, which is equal to the present market value of risk free investment, R, divided by the present market value of the entire portfolio, V.

The first term of the right-hand side of the equation below is the present market value of risky assets contained within some given portfolio and the second term of the right-hand side is the present market value of risk free investment contained within that portfolio.

$$V = V\left(w[1]\sum_s \frac{p(s)r[1,s]}{r[s]} + ... + w[n-1]\sum_s \frac{p(s)r[n-1,s]}{r[s]} \right) + R \qquad \text{(A.27)}$$

The present market value of risky investment in a given portfolio is, by the law of one price, equal to the present market value of that portfolio less the present market value of risk free investment contained within that portfolio. By comparison, R, as a manipulation of (A.25), is written below.

$$R = V \cdot w[n] \sum_s \frac{p(s)r}{r[s]} \qquad \text{(A.28)}$$

Algebra reconciling (A.26) and (A.28) produces the desired result. Q.E.D.

Proposition III: Make the assumptions of Proposition II (There exists a portfolio such that all state-contingent portfolio future values are greater than zero, and the law of one price holds; investors can short sell predicted state-contingent future values such that these that errors in these predictions are mean zero and independent of actual outcomes.) and to these add that there exists an asset which pays a risk free rate. In this context, the predicted state-contingent price relatives of assets contained within a portfolio are simple linear functions of the risk free price relative and the state-contingent price relatives of the portfolio to which they belong if and only if the equation below holds for every asset i

$$V[i] = \sum_s \frac{p(s)F[i,s]}{r[s]} \qquad \text{(A.1)}$$

Where: V [i] is the current market value of a long position in asset i, $F[i,s]$ is the future value in state s of a long position in asset i, p (s) is a joint probability describing the likelihood of a future state of the world being realized at time period one: this state obtains if and only if asset i generates a price relative r [i, s] and the given portfolio V generates a price relative r[s]. These predicted price relatives need not depend upon the ability to measure possible rates of return on the market portfolio. Other portfolios may be substituted so long as epsilon risk can be diversified away.

Proof of Necessity: the assumptions of Proposition III are sufficient to establish the existence of epsilon portfolios and, by hypothesis in this proof of necessity, predicted price relatives are simple linear functions of r and r[s]. One may then apply the definition of epsilon to a case where the predicted price relative of asset i is a simple linear function of the risk free price relative r and the state contingent price relative r[s] as follows:

$$\varepsilon[i,s] = r[i,s] - r(1 - h[i]) - h[i]r[s] \qquad \text{(A.29)}$$

Assumptions of this proposition also permit one to write the present market value of this epsilon portfolio, a quantity V (ε [i, s]):

$$V(\varepsilon[i,s]) = \sum_s \frac{p(s)r[i,s]}{r[s]} - \sum_s \frac{p(s)\hat{r}[i,s]}{r[s]} \qquad \text{(A.30)}$$

Given that prediction errors are mean zero and uncorrelated with actual price relatives, this more general result may be specialized to fit the current circumstances:

$$V(\varepsilon[i,s]) = \sum_s \frac{p(s)r[i,s]}{r[s]} - r(1-h[i])\sum_s \frac{p(s)}{r[s]} - h[i]\sum_s \frac{p(s)r[s]}{r[s]} \qquad \text{(A.31)}$$

The law of total probability and algebra requires that the third term of the right-hand side must equal h [i]. The law of one price, the assumption that state-contingent future values of a chosen portfolio V are always positive, and the assumption that there exists a risk free asset are all assumptions of this proposition and the Lemma preceding it. Invoking the Lemma requires that the second term of the right-hand side of the equation above equals (1 - h [i]). Invoking the law of one price and the assumption that state contingent future values of a chosen portfolio V are always positive requires the first term of the right-hand side of the equation above must equal:

$$1 = \sum_s \frac{p(s)r[i,s]}{r[s]} \qquad \text{(A.11)}$$

This result above could also be established by noting that the present market value of an epsilon portfolio is zero. Since for any asset i it is true by definition that every state-contingent future value of asset i is equal to the present value of asset i multiplied by the state-contingent price relative of asset i, the desired result follows. The proof of necessity is complete.

Proof of Sufficiency: to prove sufficiency, begin with the valuation of asset i, which can assumed by hypothesis --

$$V[i] = \sum_s \frac{p(s)F[i,s]}{r[s]} \qquad \text{(A.1)}$$

By definition the state-conditional future values of asset i are all equal to the present value of asset i multiplied by the state-contingent price relatives of asset i. This and some algebra provide
for the following expression:

$$1 = \sum_s \frac{p(s)r[i,s]}{r[s]} \qquad \text{(A.11)}$$

Investors may attempt to forecast the state-contingent price relatives of asset i. These predictions themselves may be bought are sold. Consider first the purchase or sale of predicted state-contingent price relatives of asset i, where these predicted price relatives are written as a Taylor series expansion. In this Taylor series expansion, the state-contingent predicted price relative on asset i in state s is written as a function of a state-contingent price relative on some portfolio V. For this Taylor series, the point r, the price relative for a risk free asset, is chosen in order to expand this function.

$$\hat{r}[i,s] = r + g'(r)(r[s]-r) + \frac{g''(r)(r[s]-r)^2}{2!} + \ldots \qquad \text{(A.32)}$$

where: $\hat{r}[i,s]$, is the predicted price relative of asset i which is contingent upon state s; r, is the price relative of the risk free asset; $r[s]$, is the price relative of portfolio V which is contingent upon state s; $g'(r)$ is the value of the first derivative of the predicted price relative of asset i with respect to a change in the price relative of portfolio V where this price relative of portfolio V is equal to the price relative of the risk free asset, similarly, $g''(r)$ is the second derivative. It is well known that a Taylor series expansion can be used to approximate any real-valued function to any degree of accuracy desired, so long as this function to be described is repeatedly differentiable. For the proof to go forward from this point, one must assume that investors may choose to predict state-contingent price relative for asset i with any Taylor series expansion equation that they would like, so the existence of higher order derivatives to accomplish this is assumed. Given the essential properties of any epsilon portfolio corresponding to asset i, the quantity V (ε [i, s]) must equal zero. This result is shown below:

$$0 = \sum_s \frac{p(s)(r[i,s] - \hat{r}[i,s])}{r[s]} \qquad \text{(A.21)}$$

Recall that by hypothesis in this proof of sufficiency that the present value of all possible price relatives for asset i is equal to unity. This present value is again shown below.

$$1 = \sum_s \frac{p(s)r[i,s]}{r[s]} \qquad \text{(A.33)}$$

Given the essential properties of an epsilon portfolio and the hypothesis of this proof of sufficiency where it is shown that the present value of all possible future price relatives of asset i is equal to unity, it must then be true that the present value of all possible predicted price relatives for asset i also is equal to unity:

$$1 = \sum_s \frac{p(s)\hat{r}[i,s]}{r[s]} \qquad \text{(A.34)}$$

So then, should an investor choose to describe possible predicted future price relatives of asset i as a Taylor series expansion, then it must still be true that the present value of these price relatives remains unity. However, as it turns out, this must be a false statement if derivatives of order two and above are non-zero.

$$\sum_s \frac{p(s)\hat{r}[i,s]}{r[s]} = \sum_s \frac{p(s)\left[r + g'(r)(r[s]-r) + \frac{g''(r)(r[s]-r)^2}{2!} + \ldots\right]}{r[s]} \qquad \text{(A.35)}$$

$$\sum_s \frac{p(s)\hat{r}[i,s]}{r[s]} = r\sum_s \frac{p(s)}{r[s]} + g'(r)\sum_s \frac{p(s)[(r[s]-r)]}{r[s]} + \frac{g''(r)}{2!}\sum_s \frac{p(s)\left[(r[s]-r)^2\right]}{r[s]} + \ldots \qquad \text{(A.36)}$$

This becomes apparent when the Lemma derived above is used to analyze both the first term and the second term of the right-hand side of the equation above. (Invoking this lemma does not require adding additional assumptions to this present argument.) Rewriting this equation after applying this Lemma produces the following equation shown below.

$$1 = 1 + g'(r)(1-1) + \frac{g''(r)}{2!}\sum_{s}\frac{p(s)\left[(r[s]-r)^2\right]}{r[s]} + \ldots \qquad \text{(A.37)}$$

It follows that for the equation above to be a true statement, the present value of all second and higher order terms must sum to zero. Thus, only predictions which are linear in r and x(s) are priced. This ends the proof of sufficiency. Proofs of both necessity and sufficiency being accomplished, this argument is complete. Q.E.D.

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