A Spreadsheet Approach to Teaching Shadow Price as Imputed Worth

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ABSTRACT

Many business students have difficulty with postoptimality analysis of linear programs. One concept that is hard to understand is that of shadow prices, especially when they are used to reflect the imputed worth of scarce resources. Students should be taught that shadow prices must be interpreted as premiums over and above the costs of resources, as reflected by the coefficients in the linear program’s objective function. The problem is compounded by the fact that some textbooks explain shadow prices incorrectly, or at least incompletely. This short paper illustrates one way in which an instructor can explain how to interpret shadow prices correctly as imputed worths. Since many students have difficulty interpreting shadow prices when only one resource constraint is involved, the analysis is restricted to this limited case. It can be extended to the case of several resource constraints, using the 100-percent rule.

INTRODUCTION

Students usually have a difficult time with linear programming. The ability to model a real world situation as a system of linear functions is a challenge that few have really had to face prior to their management science studies. Actually solving the linear program is not difficult for most, however, since most MBA classes use spreadsheet analysis for this task, and software such as the Excel Solver are quite easy to learn.

Students are usually somewhat surprised to discover that, once a linear program has been solved, their analysis of the problem has only started. Sensitivity analysis, sometimes referred to as postoptimality analysis, allows the analyst to examine the sensitivity of the model to changes in the model’s parameters. For instance, what if the analyst is uncertain about the exact values of some of the coefficients in a profit function; can the results of the model still be used with confidence? What if additional amounts of a limited resource can be obtained at a specific cost; how will the solution change? And if such a limited resource can be obtained, what is its worth? These are real world concerns, and the student must have the ability to properly interpret the results of a sensitivity analysis before he or she can really use optimization tools such as linear programming in a meaningful manner.

Having taught MBA students in a management science course over several years, the author has discovered that the most difficult aspect of sensitivity analysis to most students is related to the
interpretation of shadow prices when applied to the imputed worth of a limited resource. In fact, challenged to explicitly show in a spreadsheet how the shadow price can be used to impute the worth of a limited resource, few students have been able to do this. In this paper, a spreadsheet approach is presented to teach the concept.

EXAMPLE

As an example, suppose a company makes two products. Each product requires labor and materials to manufacture, and each product must be stored in inventory before final shipment. Table 1 contains relevant data for this problem.

<table>
<thead>
<tr>
<th></th>
<th>Product 1</th>
<th>Product 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling Price</td>
<td>$485</td>
<td>$420</td>
</tr>
<tr>
<td>Labor Hours</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Material Pounds</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Inventory Sq. Ft.</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Resource Availability</td>
<td>4,000</td>
<td>10,000</td>
</tr>
<tr>
<td>Resource Cost</td>
<td>$15.00/hr</td>
<td>$4.00/pound</td>
</tr>
<tr>
<td></td>
<td>$3.00/sq. ft.</td>
<td>900 sq. ft.</td>
</tr>
</tbody>
</table>

Table 1. Data for Example Problem

After calculating the marginal profits for each product, considering the selling price and total cost of resources for each, a linear program to maximize total profits can be written as:

\[
\text{MAX } Z = 362 X_1 + 287 X_2
\]

S.T. \[
4 X_1 + 6 X_2 \leq 4000
\]

\[
15 X_1 + 10 X_2 \leq 10000
\]

\[
X_1 + X_2 \leq 900
\]

and \[
X_1, X_2 \geq 0
\]

Where \( Z \equiv \) Total Profit, $ and \( X_i \equiv \) Number of units of Product \( i, \ i = 1, 2 \)

To solve this linear program, we first place the model in an Excel spreadsheet. Figure 1 shows this spreadsheet. Notice that all resource requirements and their costs are placed directly in the spreadsheet, so that the objective function coefficients are calculated as a function of these quantities.
This problem is easily solved using Excel’s linear programming Solver. Details of its use are not included here, but many management science textbooks include instructions. Figure 2 shows how the spreadsheet appears after the optimal values of the decision variables have been determined by the Solver. Here, it can be seen that the optimal solution calls for 400 units of product 1 and 400 units of product 2. These optimal values result in a maximum profit contribution of $259,600. Examination of the constraints reveals that all 4,000 hours of labor and 10,000 pounds of material will be utilized in this solution, and 100 square ft. of inventory space will be left over.

The shadow price of a constraint is defined as the amount of change that will result in the objective function for each unit change in the right-hand-side value of the constraint. If the constraint is relaxed by one unit, the objective function will improve by the shadow price; tightening the constraint has the opposite effect. Figure 3 contains the Excel Solver’s sensitivity
The shadow price associated with an hour of labor is $13.70. Thus, if the right-hand-side of the labor constraint is increased by one hour, the value of the objective function is increased by $13.70. How can this be illustrated? The easiest way to demonstrate this definition of shadow price is to simply change the right-hand-side value of the labor constraint, solve the problem again, and observe what happens. Figure 4 illustrates this. Here, a separate delta (Δ) column is used to show the amount of increase in the right-hand-side. Figure 4 shows the final answer after a single labor hour (Δ=1) is added, and demonstrates that the objective function indeed increases by $13.70. It is also apparent that this increase is accomplished by changing the product mix. The optimal number of units of Product 1 has decreased by 0.20, and the number of units of Product 2 has increased by 0.30.

Figure 3. Solver’s Sensitivity Report for Example Problem

Figure 4. Excel spreadsheet showing solution to LP after incrementing RHS
Now, what is the imputed worth of the labor resource? That is, what is the very most that we should be willing to pay for an additional hour of labor? Many textbooks (too many, in fact) would imply that the answer to this question is $13.70. It is not. Notice that we have been paying a variable cost of $15.00 per hour. We must interpret the imputed worth of a resource as the price we have been paying for the resource (as reflected by the coefficients in the objective function) in addition to the shadow price. The shadow price reflects a premium over and above the current cost of the resource. In this case, we could pay as much as $28.70 for an additional hour of labor and our profits would still be $259,600. Of course, we would not necessarily want to pay this amount just to keep our profits the same. However, we know that even if we paid a bonus of say, $2.00 per hour, we could still increase our profits by $11.70 ($13.70 - $2.00) for each such hour obtained.

How can this be demonstrated using a spreadsheet approach? When asked to show the effect of a $2.00-per-hour premium for labor, many students are tempted to just add $2.00 to the original cost of labor in the spreadsheet of Figure 4, and then solve the problem. They are surprised to see the results shown in Figure 5.

Notice that, instead of increasing the profit by $11.70, it has actually decreased profit by $7,988.30. Of course, the reason for this is that the $2.00 cost increase has been applied to all of the labor hours, not just the additional ones. To correctly assess the impact of the cost increase on only the additional labor hour (or hours, if $\Delta > 1$), we actually have to solve a separate linear program that deals only with these additional hours. Figure 6 shows a spreadsheet that does this. Notice that there are four “delta” columns: one for the change in the right-hand-side values, one for the change in the quantity of Product 1 to produce, one for the change in the quantity of Product 2, and one for the change in the profit. In addition, the increase in the cost of labor is shown to the right of the original cost of $15.00. This cost increase is included in the formulas for calculating the objective function coefficients for $\Delta$Product 1 and $\Delta$Product 2, which have now decreased to $354.00 and $275.00, respectively, while the coefficients for Product 1 and Product 2 remain as before ($362$ and $287$, respectively). So what we have in Figure 6 is one
Figure 6. Spreadsheet showing incremental costs, quantities and profits

linear program (the “incremental” one), resting on top of another linear program (the “original” one). Realizing that the incremental changes in the optimal values of the decision variables could be either positive or negative, this “incremental” linear program does not contain the normal non-negativity constraints. When this incremental linear program is solved, the results are as shown in Figure 7.

Figure 7. Solution to linear program in Figure 6.

Here the true picture can be seen. We have paid a total of $17.00 for the additional labor hour, and the profits have increased by $11.70, by decreasing the quantity of product 1 by 0.20 and by increasing the quantity of product 2 by 0.30. The imputed worth of the labor resource is not the shadow price alone, but the shadow price added to the original cost of $15.00. The very most we would be willing to pay for an additional hour of labor is, therefore, $28.70 ($15.00 + $13.70).
The instructor who teaches imputed worth as described in this paper should be able to dispel the students’ doubts raised by textbooks that do not explain it in the same way. For instance, Powell and Baker (2004) and Render and Stair (2003) define the imputed worth as the shadow price alone, while Taylor’s (2004) definition is more in line with the one in this paper.

The resource cost here was variable. Of course, if the cost is fixed or sunk, it would not be used in determining the coefficients of the objective function, which must be linear. In that case, the cost of the resource as reflected by the coefficients in the objective function would be zero, so the imputed worth would be the shadow price alone.

CONCLUSIONS

This paper has presented a spreadsheet approach to teaching the imputed worth of a resource as the shadow price of the resource in addition to the variable cost of the resource, as reflected by the coefficients of the linear program’s objective function. By explicitly including the original variable costs of the resources in the spreadsheet, and by applying the revised resource costs to only the incremental resource quantities, the student is unlikely to mistakenly interpret the imputed worth as the shadow price alone.

Similar conclusions can be drawn if we consider more than one resource at a time. If we apply the so-called “100 percent rule” for the changes in the different resources, the imputed worth of each resource is given by the shadow price in addition to the original variable cost of the resource. We do not illustrate this case in this paper.

REFERENCES