An Optimal Inventory Model With Partial Backorders

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ABSTRACT

Most of the inventory models discuss two extreme situations regarding the demand process when items are stock out. They are: (1) all of the demand within shortage period is backordered and (2) all of the demand within shortage period is lost sales. However, in practical inventory systems, generally, backorders and lost sales are exist with mixture simultaneously; that is, only part of the unsatisfied demand will be satisfied until the next complement of inventory. In this paper, we conduct an economic order quantity model with partial backorders, and we propose two assumptions: (1) the rate of demand within the backorder period is a decreasing function of *the shortage time. (2) Shortage will cause the opportunity cost of lost sales. The object of this research is to minimize the total relevant costs, which includes ordering cost, holding cost, backorder cost, and lost sale opportunity cost. A mathematical model is build and an efficient algorithm is developed for finding the optimal solution. We also adopt a numerical example to explain and prove this result; finally, a sensitivity analysis is performed to study the effects of the model parameters on the optimal solution. In conclusion, this paper apparently makes an improvement that avoids the discussion of extreme situations, in order to meet the applications of management practice.*

INTRODUCTION

Most of the inventory models discuss two extreme situations regarding the demand process when items are stock out. They are: (1) all of the demand within shortage period is backordered and (2) all of the demand within shortage period is lost sales. However, in real inventory system, demands during the period of stock out can be partially captive. To go a step further, if demands can be fully captive, the next replenishment will fulfill unsatisfied demands during the period of backorders. On the contrary, unsatisfied demands will be completely lost if demands can not be fully captive.

Yet, demand rate during the period of stock out is not a fix constant if to take backorders practically in consideration. That is, demand will decrease according to the increasing of the period of backorder.

In 1966, Naddor has brought up linear demand inventory model using geometry solution to determine inventory level, and used “squeezing method” to get the optimal replenishment policy. Donaldson (1977) elaborated linear demand inventory model and conduced an analytical solution. Silver (1979) developed a total relevant cost function, using differential solution to decide the optimized period to place an order.

Sachan （1984） extended and induced the concept of backorder. Also Goyal (1992) brought up an inventory model of replenishing the stock after a period of backorder, which is that deplete cycle always started from the period of backorder. Wee (1995) modified the complete backorder assumptions and proposed the concept of partial backorders, which assumed the backorder ratio is a constant between 0 and 1. The assumption is that usually the time scale of backorder will become consumers' main pondering factor to accept backorder.

Afterward, Jalan et al. (1996), Giri and Chaudhuri (1997), Chakrabarti et al. (1998) and many other scholars have developed inventory models on related field, and initiated the concept of demands will be changed through time cycle into model, also included backorder status.

Early in the research published by scholars, Fabrycky and Banks (1967) and Jelen (1970), has placed the mixed situation of backorder and lost sales in inventory model, but yet proposed the solution. Montgomery et al. (1973) proposed a series of complicated solution process in first place. Also, Rosenberg (1979) constructed a hypothetic demand rate model and proposed the solution. Park (1982) proposed only a certain percentage of demand can be replenished in backorder period, and got the result after defining backorder cost in a time based and penalty cost in every sales unit. In terms of Park's research, Padmanabhan and Vrat (1990) proposed that the demand rate in backorder status shouldn't be fixed, especially in consumption goods. Demand rate of backorder should be determined by inventory quantity of being replenished, which is an exponential function.

The organization of this paper is as follows. The assumptions and model formulation are given in the next section. The optimal solution is derived, and a solution algorithm is proposed in section 3. Then, a numerical example and sensitivity analysis of the effects of model parameters on the optimal solution are presented in section 4 and 5. The last section contains a conclusion of the results given in this paper and a discussion of possible future extended studies.

MODEL ASSUMPTION AND FORMULATION

In this paper, the demand rate is assumed as a diminishing function of the period of backorder. That is, demand will decrease complying with the incremental backorder time cycle. At the same time, diminishing demand should be considered except incremental cost of backorder. Hence, backorder cost in a fix time unit and penalty cost of lost sales should be defined respectively and added into the cost structure of the model. Park (1982) and Padmanabhan and Vrat (1990) have proposed the mixture of backorder and lost sales. This paper extends these two papers by assuming the diminishing demand rate during the period of backorder.

Assumptions

Among the process of formulating the model, a given and fixed parameter δ is added. It represents the demand lost ratio during the period of stock out, and $0 \le \delta \le 1$. Proceeded to the next step, the demand rate, $g(t)$, during the period of backorder is defined as:

$$g(t) = D(1-\delta \cdot t) \text{ and } (1-\delta \cdot t) \geq 0$$

where D is regular, fixed demand rate; t is the period of backorder, which formed demand and time related diminishing function. The figure of inventory model system is as Figure 1.

Fundamental assumption:

1. Consider single product
2. Replenish the stock instantaneously
3. Backorder is allowable
4. Demand rate of inventory depletion period is a known and fixed constant
5. Demand rate per time unit is a diminishing function during the period of backorder
6. Order cost, inventory holding cost per unit and penalty cost per unit are known and fixed constants.

Symbol exposition:

A	Order cost
D	Demand rate per unit time
H	Unit Inventory holding cost per unit time
π	Unit backorder cost per unit time
P	Penalty cost per unit of lost sales per unit time
I	Average inventory level。
B	Average backorder level。
T	Time scale of inventory cycle
t_1	Inventory depletion period（inventory level ≥ 0）
t_2	Shortage period（inventory level ≤ 0）
S^*	Uppermost inventory level
Q^*	The optimal order quantity

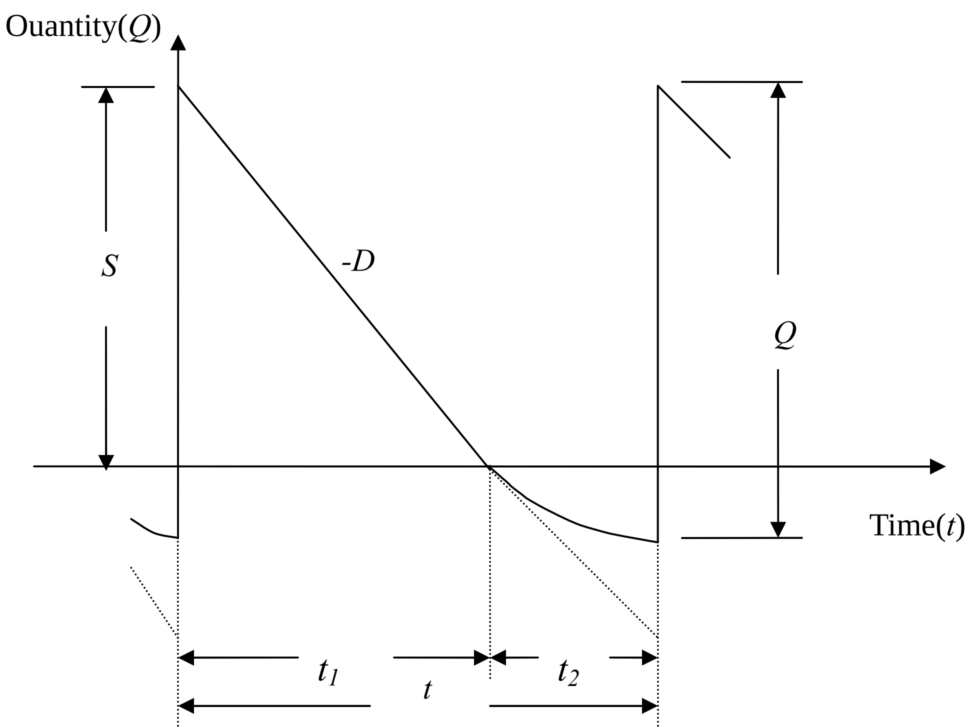

Figure 1. Inventory model with partial backorders

Besides, the total relevant cost per unit time in this research is the sum of order cost, inventory holding cost, backorder cost and penalty cost of lost sales, while material purchasing cost is not considered.

Model Formulation

Based on Fig.1, inventory cycle divided into two periods: Inventory depletion period (t_1) and backorder period (t_2). As a result in special condition, when $\delta = 0$, represents full backorders without any sales lost; when $\delta > 0$, represents lost sales and penalty cost being brought up. Hence, this research will focus on two conditions mentioned above and build up mathematical models respectively. Base on two periods of inventory status, relevant costs can be computed as below:

When diminishing demand rate $\delta = 0$

1 Order cost:

$$\frac{A}{T} = \frac{A}{t_1 + t_2} \tag{1}$$

2 Holding cost:

During an inventory cycle, inventory holding cost only take place in t_1, and the average inventory level is as below:

$$I = \frac{Dt_1^{\ 2}}{2} \times \frac{1}{t_1 + t_2} = \frac{Dt_1^{\ 2}}{2(t_1 + t_2)}$$

According to the function above, the inventory holding cost is:

$$hI = \frac{hDt_1^2}{2(t_1 + t_2)} \tag{2}$$

3 Backorder cost:

Since backorder can be fully captive during the period of t_2, which means demand rate is still a fixed constant. Hence, the average backorder level is:

$$B = \frac{Dt_2^2}{2(t_1 + t_2)}$$

Therefore, backorder cost is:

$$\pi B = \frac{\pi D t_2^2}{2(t_1 + t_2)} \tag{3}$$

Let $TRC_1(t_1,t_2)$ represent total relevant cost function of inventory model, which is the sum of (1), (2) and (3):

$$TRC_1(t_1, t_2) = \frac{A}{t_1 + t_2} + \frac{hDt_1^2}{2(t_1 + t_2)} + \frac{\pi D t_2^2}{2(t_1 + t_2)} \tag{4}$$

The optimal t_1 and t_2 can be obtained as below:

$$t_1^* = \sqrt{\frac{2A}{Dh(1 + \frac{h}{\pi})}} \tag{5}$$

$$t_2^* = \sqrt{\frac{2Ah}{D\pi(\pi + h)}} \tag{6}$$

Using t_1* and t_2* to calculate the highest inventory level S* and the optimal order quantity Q* as below:

$$S^* = D\, t_1^*$$

$$Q^* = D(t_1^* + t_2^* - \frac{\delta \cdot (t_2^*)^2}{2})$$

When diminishing demand rate $\delta > 0$

（1）Order cost:

$$\frac{A}{T} = \frac{A}{t_1 + t_2} \tag{7}$$

（2）Holding cost:

In every inventory cycle, inventory holding cost only happen during t_1. Hence, the average stock level is as below:

$$I = \frac{Dt_1^2}{2} \times \frac{1}{t_1 + t_2} = \frac{Dt_1^2}{2(t_1 + t_2)}$$

And the inventory holding cost is:

$$hI = \frac{hDt_1^2}{2(t_1 + t_2)} \tag{8}$$

(3) Backorder cost:

Backorder happens in the period of t_2. Since the demand rate is a decreasing function of backorder period, defined as $g(t) = D(1-\delta t)$, which is the rate of change of demand, the slope of the function. Hence, the backorder function at any time t during the stock out period can be obtained by integrating $g(t)$. The maximum backorder level at time t_1+t_2 can be obtained:.

$$\int_0^{t_2} g(t)dt = \int_0^{t_2} D(1 - \delta \cdot t)dt = Dt_2 - \frac{D\delta \cdot t_2^{\ 2}}{2}$$

Let $b(t) = Dt - \frac{D\delta \cdot t^2}{2}$ represent backorder function during the period of t_2 and the total quantity of backorder is

$$\int_0^{t_2} b(t) = \int_0^{t_2} Dt - \frac{D\delta \cdot t^2}{2} dt = \frac{Dt_2^{\ 2}}{2} - \frac{D\delta \cdot t_2^{\ 3}}{6}$$

Therefore, the average backorder level is:

$$B = \left(\frac{Dt_2^{\ 2}}{2} - \frac{D\delta \cdot t_2^{\ 3}}{6} \right) \times \frac{1}{t_1 + t_2} = \frac{Dt_2^{\ 2}}{2(t_1 + t_2)} - \frac{D\delta \cdot t_2^{\ 3}}{6(t_1 + t_2)}$$

and the backorder cost is:

$$\pi B = \left(\frac{Dt_2^{\ 2}}{2} - \frac{D\delta \cdot t_2^{\ 3}}{6} \right) \times \frac{1}{t_1 + t_2} = \frac{\pi Dt_2^{\ 2}}{2(t_1 + t_2)} - \frac{\pi D\delta \cdot t_2^{\ 3}}{6(t_1 + t_2)} \tag{9}$$

（4）Penalty cost:

Penalty cost of lost sales caused by backorder is:

$$p\left[\frac{Dt_2^{\,2}}{2}-\left(\frac{Dt_2^{\,2}}{2}-\frac{D\delta\cdot t_2^{\,3}}{6}\right)\right]\times\frac{1}{t_1+t_2}=\frac{pD\delta\cdot t_2^{\,3}}{6(t_1+t_2)} \tag{10}$$

Let $TRC_2(t_1,t_2)$ represent the function of total relevant cost. It is the sum of (7), (8), (9), and (10):

$$TRC_2(t_1,t_2)=\frac{A}{t_1+t_2}+\frac{hDt_1^{\,2}}{2(t_1+t_2)}+\frac{\pi Dt_2^{\,2}}{2(t_1+t_2)}-\frac{\pi D\delta\cdot t_2^{\,3}}{6(t_1+t_2)}+\frac{pD\delta\cdot t_2^{\,3}}{6(t_1+t_2)} \tag{11}$$

OPTIMAL SOLUTION AND SOLUTION ALGORITHM

Optimal Solution

When $\delta > 0$, the necessary conditions to optimize the target function TRC_2 are:

$$\frac{\partial TRC_2}{\partial t_1}=0 \quad \text{and} \quad \frac{\partial TRC_2}{\partial t_2}=0$$

The result of the calculation is:

$$t_1=\frac{2\pi\cdot t_2+\delta(p-\pi)t_2^{\,2}}{2h} \tag{12}$$

Another relationship between t_1 and t_2 in (12) is:

$$t_2=\frac{-\pi+\sqrt{\pi^2+2h\delta(p-\pi)t_1}}{\delta(p-\pi)} \tag{13}$$

When $\delta = 0$, t_1 and t_2 can be obtained by substituting given parameter values into (5) and (6); when $\delta > 0$, the relationship between t_1 and t_2 can be obtained from (12). However, the relationship is not a closed form. As result, an algorithm needs to be developed to find the solutions.

Solution Algorithm

When $\delta > 0$, the solution can not be found directly and have to use proper one dimension search to get the results. The golden section search to used in this research to decide the value t_2 then determine the value of t_1 and calculate total relevant cost. Therefore, the assumption of demand change rate in backorder status in this model is $g(t) = D(1-\delta t)$. Since the backorder rate is nonnegative, that is $(1\text{-}\delta{\cdot}t) \geq 0$. Therefore, the range of t_2 is $\left[0, \frac{1}{\delta}\right]$.

The search algorithm is as follow:

(1) For a given t_2 value, substitutes it into (12) and find t_1.

(2) Substituted t_1 and t_2 into total relevant cost function (11).

(3) Give different t_2 value and repeat previous steps until the minimum total relevant cost and optimal t_1 and t_2 is found.

A NUMERICAL EXAMPLE

A numerical example is used to illustrate the solution algorithm presented above. This example is also used in the sensitivity analysis in the next section.

Assumed the value of each parameter is as follow:

Demand lost rate $\delta = 0.3$

Order cost, A = \$50 / each

Demand rate, D = 100 units / unit time ◦

Inventory holding cost per unit, h = \$ 0.1 / unit time ◦

Backorder cost per unit, π = \$0.3 / unit time ◦

Penalty cost of lost sales per unit, p = \$0.4 / unit time ◦

Values of t_1, t_2, Q^*, S^*, and total relevant cost (TRC) can be found after substituting the examples above to the model and divided total relevant cost into 4 parts for observing in sensitivity analysis: order cost (A_0), holding cost (h_0), backorder cost (π_0), and penalty cost (p_0)

δ	t_1	t_2	TRC	Q^*	A_0	h_0	π_0	p_0
0.3	**2.73**	**0.87**	**27.48**	**349.33**	**13.86**	**10.36**	**2.89**	**0.37**

Table 1. Numerical solution of the example

Golden section search can find that when $t_2 \leq \frac{1}{0.3}$ known from Table 1, the optimal backorder period is $t_2 = 0.87$, substituted (12) can get $t_1 = 2.73$ and when t_1 and t_2 substituted into (11) can get total relevant cost TRC = 27.48. At the time, the optimal order quantity is Q^* = 349.33units. The total relevant cost includes order cost (A_0) is 13.86, holding cost (h_0) is 10.36, backorder cost

(π_0) is 2.89 and penalty cost (p_0) is 0.37 .

SENSITIVITY ANALYSIS (Omitted)

CONCLUSION

To conclude the research:

（1）After sensitivity analysis can induce the following findings:

a. When δ increase, the optimal order quantity will decrease, but total relevant cost will increase. This deduction is in chorus with Padmanabhan and Vrat (1990)'s deduction.

b. When replenishing cost A and demand rate per unit time D increase, order quantity Q^* and total relevant cost (TRC) will increase.

c. Any of holding cost per unit (h), backorder cost (π) and penalty cost (p) increase will make total relevant cost (TRC) increase, but optimal order quantity Q^* will decrease

d. The increment of inventory holding cost per unit (h), backorder cost (π) and penalty cost (p) will lead to the phenomenon of increasing before diminishing to these cost items, h_0, π_0, and $p_{0,}$ in the total relevant cost function. This phenomenon can induce cost items in inventory depletion period having a trade-off relationship with cost items in backorder status, π_0 and $p_{0.}$ This relationship is the main force to minimize the cost.

（2）This essay mainly extends two researches, which investigated the assumption of demand change rate of backorder from Park (1982) and Padmanabhan and Vrat (1990). Both considered that only part of demand being satisfied, the rest would induce lost sales. Park (1982) assumed that diminishing demand rate was fixed and irrelevant to the time scale, and Padmanabhan and Vrat (1990) assumed that diminishing demand rate was exponent to the time scale. This essay assumed the diminishing demand rate is a diminishing function to backorder status, which is significantly different to the basic assumption in previous.

（3）When $\beta = 0$and $\beta = 1$ in Park's (1982) model, represented two extreme cases, total lost and total replenishment in backorder status respectively. When $\delta = 0$ in Padmanabhan and Vrat's (1990) model, represented demands in backorder status can be fully replenished. But in this essay, during backorder status t_2, as （13）, found δ as denominator in the model, which means it is meaninglessness when $\delta = 0$. Hence, has to put $\delta = 0$ and $\delta > 0$ into to situations. The main purpose of this research is to probe that only part of demand during backorder status can be satisfied ($\delta > 0$). Therefore, $\delta = 0$, full replenishment is not the issue. Compare with models from Park (1982) and Padmanabhan and Vrat (1990), the result of this research is unanimity to its assumption.

But in order to avoid discuss under the extreme situation, this research induced from Park (1982) and Padmanabhan and Vrat (1990) and improved the process of getting result.

This research is to build EOQ model without considering production factor. In the future, production factor will be added to build EPQ model in order to probe the influence of the assumption of diminishing demand rate in backorder status.

REFERENCE (omitted)

Full paper is available upon request.