Excel-Based Solution Method For The Optimal Policy Of The Hadley And Whittin's Exact Model With Arma Demand

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ABSTRACT

In the determination of the optimal policy of an inventory model with a stochastic demand which includes the calculation of the reorder point and the order size, one has to deal with mean rate of demand, standard deviation, safety factor, forecast and lead-time. The calculation of the re-order point is typically based on the assumption that the mean rate demand is deterministic as a function of time. This assumption is by far removed from reality. A more appropriate assumption would involve the use some sort of probability distributions to represent the units demanded including the lead-time to account for the increasing uncertainty in the market environment. Under the stochastic environment, Hadley and Whitten developed two types of backorder inventory formulas; an approximate and exact formulae for Poisson and Normal lead-time demand distributions with the assumption that there is no correlation between two period demands. In many practical situations, the period demands are not independent, but exhibit a serially correlated process. (An, Fotopolo, and Wang 1989); (Charles, Marmorsten, and Zinn 1995). In this research paper, we will develop the formula for the calculation of reorder point, safety stock and order quantity of Hadley and Whittin's (1963) exact inventory model when the units demanded are generated by a serially correlated process and can be represented by ARMA Box-Jenkins time series model for when the lead-time is both deterministic and probabilistic. ARMA time series process generating demand with deterministic and stochastic discrete lead-time.

The distribution of forecast errors from the calculation process in Box-Jenkins' (1976) ARMA analysis will be used as the measurement of the estimation with which the reorder point and safety stock are determined. In the first part of this research, the determination of the model's reorder point is based on the assumption that the procurement lead-time is a random variable generated by an ARMA process with constant lead times. Later on, we would investigate the problem in attempting to account for the ARMA system with the probabilistic discrete lead-times.

I. INTRODUCTION

The control and maintenance of inventories of physical goods is a problem common to all enterprises in any given economy. Two fundamental questions that must be answered in controlling the inventory of any physical goods are when to replenish the inventory and how much to order for replenishment. EOQ models answers the question of how much to order, but not the question of when to order. The latter is the function of models that identify the reorder in terms of a quantity: the reorder point occurs when the quantity on hand drops to a predetermined amount. The amounts generally includes expected demand during lead time and perhaps an extra cushion of stock, which serve to reduce the risk of experience a stock-out during lead time especially in the environment when variability is present in demand or lead time or both. The following four factors are being used in determining the reorder point quantity

1. The rate of demand (usually based on a forecast value).

2. The length of lead-time.

3. The variability of demand/or lead-time.

4. The degree of acceptable stock-out risk.

Taking into the consideration of these four factors, Hadley and Whittin (1963) suggested both approximate and exact <Q,r> models with backorder which attempts to answer both two fundamental questions mentioned above. Their expected costs included in the model are, the expected annual setup, holding, and the shortage costs. Under the normal distribution environment, the average annual cost is

$$K = \frac{D}{Q}A + IC[\frac{Q}{2} + r - \mu]\ \pi E(Q,r) + (\bar{\pi} + IC)B(Q,r)$$

where

D = Average annual units demanded

Q = Order quantity

A = Cost per order

I = Carrying charge in dollars per dollar per year

C = Unit cost of the inventory

r = Reorder point

μ = Average lead-time demand

π = Backorder cost in dollars per backorder

$\bar{\pi}$ = Shortage cost in dollars per unit year of shortage

$E(Q,r)$ = The expected number of backorder incurred per year

$$= \frac{D}{Q}[\alpha(r) - \alpha(r+Q)]$$

$$\alpha(v) = \sigma\phi\left(\frac{v-\mu}{\sigma}\right) - (v-\mu)\Phi\left(\frac{v-\mu}{\sigma}\right)$$

$\phi(*) =$ The normal density function

$\Phi(*) =$ The complementary cumulative of the normal distribution

$B(Q,r) =$ The expected number of backorders at any time

$$= \frac{1}{Q}[\beta(r) - \beta(r+Q)]$$

$$\beta(v) = 0.5[\sigma^2 + (v-\mu)^2]\Phi\left(\frac{v-\mu}{\sigma}\right) - 0.5\sigma(v-\mu)\phi\left(\frac{v-\mu}{\sigma}\right)$$

II. DETERMINING THE MEAN AND VARIANCE OF THE LEAD-TIME DEMAND

In order to compute the reorder point with a safety stock that will meet a specific service level, we have to know the probability density of the lead time demand, the sum demand during the lead time period, and the variance of the total lead time demand.

When the demand can be represented by an ARMA process [Box et al, 1976], the conditional probability distribution $p(z_{t+l} \mid z_t, z_{t-1}, ... z_1)$ of the future value z_{t+l} of the process will be Normal with mean $\hat{z}_t(l)$ - the forecast of the future z_{t+l} from the origin t, and variance $\{1 + \sum_{j=1}^{l-1} \psi_j^2\}\sigma_a^2$ and then $p(z_{t+l}, z_{t+l-1}, ... z_{t+1} \mid z_t, z_{t-1}, ... z_1)$ is a multivariate with mean

$$\hat{Z}_t = \begin{bmatrix} \hat{z}_t(1) \\ \\ \\ \hat{z}_t(l) \end{bmatrix},$$

where $\hat{z}_t(l)$ is the forecast value of z_{t+l} provided that $z_t, z_{t-1}, ... z_1$ values are available, and the covariance matrix

$$G = \sigma_a^2 \begin{bmatrix} g_{11} & g_{1l} \\ g_{21} & g_{2l} \\ & \\ g_{l1} & g_{ll} \end{bmatrix}$$

where $g_{jj} = \{1 + \sum_{j=1}^{l-1} \psi_j^2\}$ and $g_{l,l+j} = \sum_{i=0}^{l-1} \psi_i \psi_{j+i}$, where $\psi_0 = 1$.

The total amount of demand during the lead-time period is

$S_t = z_{t+l} + z_{t+l-1} + \ldots\ldots\ z_{t+1} = UZ_t$, where $U = [1, 1, \ldots\ldots\ .1, 1]$ and

$$Z_t = \begin{bmatrix} z_{t+l} \\ z_{t+l-1} \\ \\ z_{t+2} \\ z_{t+1} \end{bmatrix} \text{ and } E(Z_t) = \begin{bmatrix} \hat{z}_t(l) \\ \\ \\ \\ \hat{z}_t(1) \end{bmatrix} = \hat{Z}_t$$

$$\mathrm{E}(S_t) = UE(Z_t) = U\hat{Z}_t = \hat{z}_t(l) + \hat{z}_t(l-1) + \ldots\ldots + \hat{z}_t(1)$$

$$Var\ (S_t) = UZ_t Z_t^T U^T = \sigma_a^2 \sum_{i=1}^{l} \sum_{j=1}^{l} g_{ij}$$

As we can see from the above analysis that, for Gaussian demand like ARMA process, the problem reduces to identifying the first two moments of the distribution of the demand rate for each period during the lead-time period.

The following steps will be used to compute the variance of a given lead-time.

1. Calculating of the ψ_j weights using the following equations:

   $$\psi_1 = \varphi_1 - \theta_1 \quad \text{j}$$
   $$\psi_2 = \varphi_1 \psi_1 + \varphi_2 - \theta_2$$
   $$\psi_j = \varphi_1 \psi_{j-1} + \ldots\ldots + \varphi_{p+d} \psi_{j-p-d} - \theta_j$$

   where $\psi_0 = 1, \psi_j = 0\ for\ \ j < 0\ \ and\ \ \theta_j = 0\ for\ \ j > q$. and φ_j and θ_j are the coefficients of the autoregressive and moving average in ARMA

2. Calculating g_{ij} and g_{ii}.

3. Compute $\hat{z}_t(i)$, for $i = 1, \ldots\ldots\ l$, the forecast values using the difference equation forms and then compute $E(S_t) = \hat{z}_t(l) + \hat{z}_t(l-1) + \ldots\ldots + \hat{z}_t(1)$

4. Compute $Var(S_t) = \sigma_a^2 \sum_{i=1}^{l} \sum_{j=1}^{l} g_{ij}$

See Appendix I – the Excel template for the computation of the mean and variance of the forecast error distribution.

Example. Suppose that the lead-time demand can be represented by an ARMA(2,2) model as

$$Z_t - 1.6Z_{t-1} + 0.62Z_{t-2} = a_t - 0.82a_{t-1} + 0.42a_{t-2}$$

Using the Excel Template in Exhibit I, the value of standard deviation of the lead-time demand = 51.47883., for $\sigma_a = 5.78$.

III. SOLUTION COMPUTATION METHOD BY SOLVER (SEE APPENDIX II)

According to Hadley and Whitten's (1963) analysis, the terms $\alpha(r+Q)$ and $\beta(r+Q)$ are negligible in the usual case. Thus for a given value of reorder point r, the optimal value of Q can be determined from the following formula

$$Q = \sqrt{\frac{2DA(r)}{IC}}$$

where $A(r) = A + \pi\alpha(r) + 2(\frac{\bar{\pi} + IC}{D})\beta(r)$ and the average total cost for a given value of r is

$K(r) = \sqrt{2DA(r)IC} + IC(r-\mu)$.

If lead-time periods are treated as discrete random variables as suggested by Boone et al (2000), then our expected total cost of the model can easily modified to incorporate the probabilities of the time periods as follows.

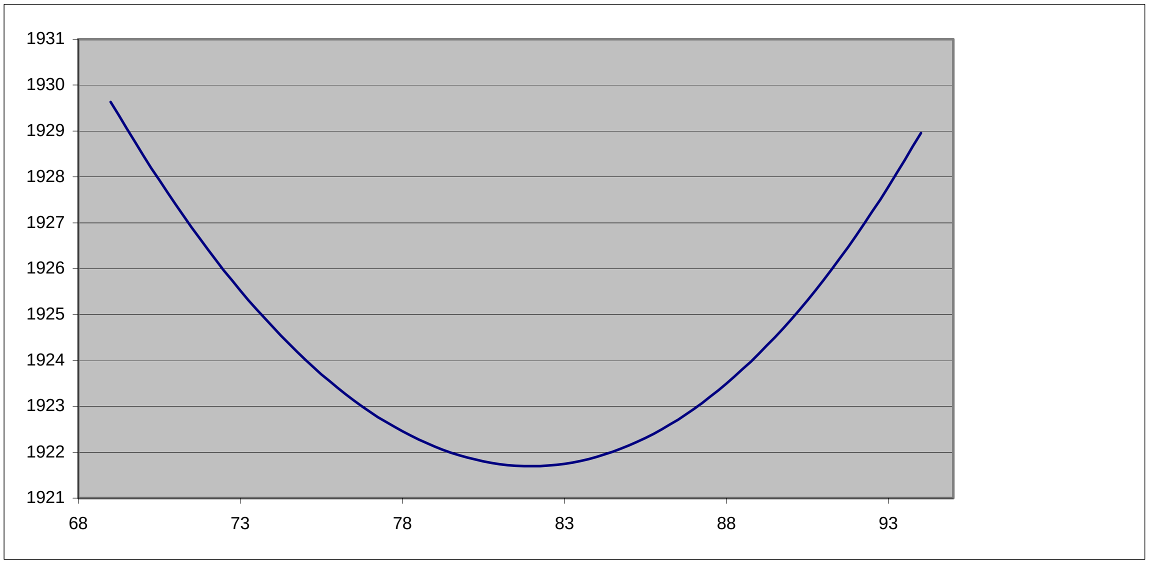
$K_p(r) = \sqrt{2DA_p(r)IC} + IC(r-\mu_p)$, where

$$A_p(r) = \pi\alpha_p(r) + 2(\frac{\bar{\pi} + IC}{D})\beta_p(r)$$

$$\alpha_p(v) = \sum_{L=1}^{M}\sigma_L\phi\left(\frac{v-\mu_L}{\sigma_L}\right)p_L - \sum_{L=1}^{M}(v-\mu_L)\Phi\left(\frac{v-\mu_L}{\sigma_L}\right)p_L$$

$$\beta_p(v) = 0.5\sum_{L=1}^{M}[\sigma_L{}^2 + (v-\mu_L)^2]\Phi\left(\frac{v-\mu_L}{\sigma_L}\right)p_L - 0.5\sum_{L=1}^{M}\sigma_L(v-\mu_L)\phi(\frac{v-\mu_P}{\sigma_L})\,p_L$$

and p_L here is the probability that there are L periods in the lead-time of the model.



The above figure shows the average total cost curve of the following parameters.

D = 700 units per year
C = $50.00 per unit of the inventory
I = $0.20 per dollar per year
A = $15.00 per order
π = $1.00 per backorder
$\bar{\pi}$ = $15.00 per unit year of shortage

Using the Solver, the optimal solution is $r^* = 81.97757, Q^* = 410.19524, K(r^*) = 1921.688$. See Appendix II.

REFERENCES.

Box, George E. P. Box and Jenkins, Jenkins, Gwilym, M., (1976). Time Series Forecasting and Control., Holden-Day, San Francisco.

Hadley, G., and Whitten, T.,M. (1963). Analysis of Inventory Systems, Prentice-Hall., Englewood Cliffs, N.J.

Boone, Tonya, and Ganeesham, Ram, (2000). "Models and Methods to Support a New Type of Inventory Performance Measure: The ESWSO, Decision Science, Vol. 31, No. 1, Winter 2000.

Appendix Ia

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1				**Infinite Form Factor**														
2	J	**AR-Factor**		Si0	Si1	Si2	Si3	Si4	Si5	Si6	Si7	Si8	Si9	Si10	Si11	Si12	Si13	Si14
3	1	1.8		1	1.8	0	2.952	3.3616	3.689	3.951	4.1611	4.3289	4.463	4.5705	4.6564	4.7251	4.7801	4.82408
4	2	-0.8			1	1.8	2.44	2.952	3.362	3.689	3.9514	4.1611	4.329	4.4631	4.5705	4.6564	4.72512	4.7801
5	3	0			0	1	1.8	2.44	2.952	3.362	3.6893	3.9514	4.161	4.3289	4.46313	4.5705	4.6564	4.72512
6	4	0			0	0	1	1.8	2.44	2.952	3.3616	3.6893	3.951	4.1611	4.32891	4.4631	4.5705	4.6564
7	5	0			0	0	0	1	1.8	2.44	2.952	3.3616	3.689	3.9514	4.16114	4.3289	4.46313	4.5705
8	6	0			0	0	0	0	1	1.8	2.44	2.952	3.362	3.6893	3.95142	4.1611	4.32891	4.46313
9	7	0			0	0	0	0	0	1	1.8	2.44	2.952	3.3616	3.68928	3.9514	4.16114	4.32891
10	8	0			0	0	0	0	0	0	1	1.8	2.44	2.952	3.3616	3.6893	3.95142	4.16114
11	9	0			0	0	0	0	0	0	0	1	1.8	2.44	2.952	3.3616	3.68928	3.95142
12	10						0	0	0	0	0	0	1	1.8	2.44	2.952	3.3616	3.68928
13	11						0	0	0	0	0	0	0	1	1.8	2.44	2.952	3.3616
14	12						0							0	1	1.8	2.44	2.952
15	13	0			0	0	0							0	0	1	1.8	2.44
16	14	0			0	0	0							0	0	0	1	1.8
17	15	0			0	0	0	0	0	0	0	0	0	0	0	0	0	1
18	16	0			0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	17	0			0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	18	0			0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	19	0			0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	20	0			0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	21	0			0	0	0	0	0	0	0	0	0	0	0	0	0	0
24															0	0	0	0
25					0	0	0	0	0	0	0	0	0	0	0	0	0	0

=SUMPRODUCT($B$3:$B$23,D3:D23)-E$25,then Copy to O3.

=D3, then copy to O3 by dragging.
Highlight D:O3, then copy by dragging to O23.

Appendix Ib

MA-Factor				Theta1	Theta2	Theta3	Theta4	Theta5	Theta6	Theta7	Theta8	Theta9	Theta10	Theta11	Theta12	Theta13	Theta1
Lead Time				1	2	3	4	5	6	7	8	9	10	11	12	13	14
	Phi																
Lead Ti	1		1														
1	1.8		1.8														
2	2.44		2.44														
3	2.952		2.952														
4	3.3616		3.3616														
5	3.68928		3.6893														
6	3.951424		3.9514														
7	4.161139		4.1611														
8	4.328911		4.3289														
9	4.463129		4.4631														
10	4.570503		4.5705														
11	4.656403		4.6564														
12	4.725122		4.7251														
13	4.780098		4.7801														
14	4.824078		4.8241														

	G
13	=SUMPRODUCT($B$3:$B$23,E3:E23)-F$25

Appendix Ic

=SUMPRODUCT(OFFSET(Phi,ABS(B$48-$A49),0,MIN(B$48,$A49)),OFFSET(Phi,0,0,MIN(B$48,$A49)))

	L + J													
L	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1	1	1.8	2.44	2.952	3.3616	3.6893	3.9514	4.161	4.329	4.4631	4.5705	4.656	4.7251	4.7801
2	1.8	4.24	6.192	7.754	9.0029	10.002	10.802	11.44	11.95	12.363	12.69	12.95	13.162	13.3293
3	2.44	6.192	10.194	13.39	15.956	18.005	19.644	20.96	22	22.843	23.515	24.05	24.481	24.8251
4	2.952	7.754	13.395	18.91	23.318	26.847	29.669	31.93	33.73	35.179	36.335	37.26	38	38.5922
5	3.3616	9.003	15.956	23.32	30.208	35.72	40.13	43.66	46.48	48.737	50.543	51.99	53.144	54.069
6	3.68928	10	18.005	26.85	35.72	43.819	50.298	55.48	59.63	62.945	65.599	67.72	69.421	70.7794
7	3.951424	10.8	19.644	29.67	40.13	50.298	59.433	66.74	72.59	77.264	81.005	84	86.393	88.3088
8	4.161139	11.44	20.955	31.93	43.657	55.481	66.741	76.75	84.75	91.158	96.282	100.4	103.66	106.284
9	4.328911	11.95	22.004	33.73	46.48	59.628	72.587	84.75	95.49	104.07	110.94	116.4	120.84	124.353
10	4.463129	12.36	22.843	35.18	48.737	62.945	77.264	91.16	104.1	115.41	124.47	131.7	137.53	142.17
11	4.570503	12.69	23.515	36.34	50.543	65.599	81.005	96.28	110.9	124.47	136.3	145.8	153.32	159.376
12	4.656403	12.95	24.052	37.26	51.988	67.722	83.999	100.4	116.4	131.73	145.76	158	167.76	175.58
13	4.725122	13.16	24.481	38	53.144	69.421	86.393	103.7	120.8	137.53	153.32	167.8	180.31	190.344
14	4.780098	13.33	24.825	38.59	54.069	70.779	88.309	106.3	124.4	142.17	159.38	175.6	190.34	203.155

Appendix II - Solver solution

D	A	IC	PI	Pih	Mu	Sd				
700	15	10	1	15	300	51.47883				
r	**Z**	**Density**	**Complement**	**Alpha**	**Beta**	**A(r)**	**K(r)**	**Q (r)**		
150	-2.9138	8.864E-05	0.9982	149.7368	12552.9276	613.0556	1429.637973	292.9637973		
160	-2.7196	1.583E-04	0.9967	139.5506	11089.2438	550.5950	1376.387914	277.6387914		
170	-2.5253	2.717E-04	0.9942	129.2626	9719.4459	491.3857	1322.860925	262.2860925		
180	-2.3311	4.479E-04	0.9901	118.8380	8442.2320	435.3463	1268.774684	246.8774684		
190	-2.1368	7.095E-04	0.9837	108.2427	7256.7788	382.4134	1213.825375	231.3825375		
200	-1.9425	1.080E-03	0.9740	97.4520	6163.1393	332.5642	1157.753055	215.7753055		
210	-1.7483	1.579E-03	0.9598	86.4627	5162.5802	285.8405	1100.441847	200.0441847		
220	-1.5540	2.218E-03	0.9399	75.3072	4257.7035	242.3680	1042.051092	184.2051092		
230	-1.3598	2.995E-03	0.9131	64.0677	3452.1931	202.3603	983.16495	168.316495		
240	-1.1655	3.884E-03	0.8781	52.8858	2750.0824	166.1030	924.9399487	152.4939949		
250	-0.9713	4.839E-03	0.8343	41.9638	2154.5640	133.9125	869.2244059	136.9224406		
260	-0.7770	5.794E-03	0.7814	31.5553	1666.5230	106.0740	818.6204406	121.8620441		
270	-0.5828	6.664E-03	0.7200	21.9423	1283.1250	82.7682	776.4546827	107.6454683		
280	-0.3885	7.365E-03	0.6512	13.4028	996.8644	64.0051	746.6102511	94.66102511		
290	-0.1943	7.821E-03	0.5770	6.1727	795.4243	49.5807	733.1448406	83.31448406		
300	0.0000	7.979E-03	0.5000	0.4107	662.5175	39.0721	739.6006526	73.96006526		
310	0.1943	7.821E-03	0.4230	-3.8273	579.6106	31.8731	767.9996078	66.79996078		
320	0.3885	7.365E-03	0.3488	-6.5972	528.1706	27.2660	817.8382061	61.78382061		
330	0.5828	6.664E-03	0.2800	-8.0577	491.9099	24.5105	885.7876074	58.57876074		
340	0.7770	5.794E-03	0.2186	-8.4447	458.5119	22.9307	966.5952997	56.65952997		
350	0.9713	4.839E-03	0.1657	-8.0362	420.4710	21.9806	1054.73323	55.47332299		
360	1.1655	3.884E-03	0.1219	-7.1142	374.9526	21.2769	1145.781064	54.57810638		
370	1.3598	2.995E-03	0.0869	-5.9323	322.8419	20.5978	1236.999651	53.6999651		
380	1.5540	2.218E-03	0.0601	-4.6928	267.3314	19.8547	1327.224979	52.72249792		
390	1.7483	1.579E-03	0.0402	-3.5373	212.4548	19.0504	1416.434798	51.64347984		
400	1.9425	1.080E-03	0.0260	-2.5480	161.8957	18.2340	1505.248907	50.5248907		
	Reod	**Z**	**Density**	**Complement**	**Alpha**	**Beta**	**Ar**	**K(r)**	**Q (r)**	
Optimal	291.9681	-0.15602	1.19263E-10	0.56199	4.51386	762.7872	46.7563	728.747068	80.9066	
						DensityP	**DensityPP**	**ComplementP**	**ComplementPP**	
						3.61465E-13	1.13931E-14	2.31674E-12	7.02E-15	
AlphaP	**AlphaPP**	**BetaP**	**BetaPP**	**ArP**	**ArPP**	**Second**	**Factor**	**SecondD**	**ConvexZero**	
-0.56199	-0.56199	-4.51386	0.561992588	-0.72320185	-0.54192142	-51.19945937	0.092521864	-4.737069414	4.737069	
							CriticalV	**Abs(Critical)**		
							3.742891778	3.742891778		